MATH2050C Assignment 6

Deadline: Feb 26 , 2018.

Hand in: 3.4 no 7; 3.5 no 3, 5, 9; Supp Ex no. 1.

Section 3.4 no. 4, 7, 9, 11.

Section 3.5 no. 2, 3, 4, 5, 9.

Supplementary Exercises

1. Can you find a sequence from $[0, 1]$ with the following property: For each $x \in [0, 1]$, there is subsequence of this sequence taking x as its limit? Suggestion: Consider the rational numbers.

2. The concept of a sequence extends naturally to points in $\mathbb{R}^N$. Taking $N = 2$ as a typical case, a sequence of ordered pairs, $\{\mathbf{a}_n\}, \mathbf{a}_n = (x_n, y_n)$, is said to be convergent to $\mathbf{a}$ if, for each $\varepsilon > 0$, there is some n_0 such that

$$|\mathbf{a}_n - \mathbf{a}| < \varepsilon \ , \quad \forall n \geq n_0 \ .$$

   Here $|\mathbf{a}| = \sqrt{x^2 + y^2}$ for $\mathbf{a} = (x, y)$. Show that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

3. Bolzano-Weierstrass Theorem in $\mathbb{R}^N$ reads as, every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence. Prove it. A sequence is bounded if $|\mathbf{a}_n| \leq M$, $\forall n$, for some number M.

4. The Fibonacci sequence is defined by $f_{n+1} = f_n + f_{n-1}, f_1 = f_2 = 1$. Consider the sequence $\{a_n\}$ given by $a_n = f_n/f_{n+1}$. Establish the followings:

   (a) $1/2 \leq a_n \leq 1$.

   (b) $\{a_n\}$ is a Cauchy sequence.

   (c) Find the limit of $\{a_n\}$.

   Hint: Observe $a_{n+1} = 1/(1 + a_n)$.

Essential 3.4 Bolzano-Weierstrass Theorem

In this section you should know

- Definition of the subsequence of a sequence. When a sequence is convergent, all its subsequences are convergent to the same limit. Consequently, a sequence is divergent if it has two convergent subsequences with different limits.
- Bolzano-Weierstrass Theorem.
- Definition of a limit point. A bounded sequence is convergent if and only if its limit point is unique.
- Definitions of the limit superior and limit inferior of a bounded sequence.

The main result in this section is:

Theorem 6.1 (Bolzano-Weierstrass Theorem). Every bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}$ be a bounded sequence. Fix a closed, bounded interval I_0 containing the sequence. We divide I_0 equally into two closed, subintervals. Since the sequence has infinitely a_n's, one of these subintervals must contain infinitely many of them. Pick and call it I_1. Next, we divide I_1 equally into two subintervals and apply the same principle to pick I_2. Repeating this process, we end up with closed intervals $I_k, k \geq 1$, with the properties: For $k \geq 1$, (a) $I_{k+1} \subset I_k$, (b) the length of I_{k+1} is half that of I_k, and (c) there are infinitely entries from $\{a_n\}$ sitting inside each I_k. Applying Nested Interval Theorem, $\cap_{k=1}^{\infty} I_k = \{\xi\}$. Now, we pick a subsequence $\{b_k\}$ from each I_k to form a subsequence. This is possible because there are infinitely many a_n's in each I_k. Clearly, $\{b_k\}$ converges to ξ.

You may use this proof to replace the proof in Text.

A point a is called a **limit point** of the sequence $\{a_n\}$ if it is the limit of a subsequence of $\{a_n\}$. A bounded sequence has at least one limit point according to Bolzano-Weierstrass Theorem. A properly divergent sequence does not have any limit point.

Theorem 6.2. A bounded sequence is convergent if and only if it has a unique limit point.

Proof. The only if part is obvious. We only consider the if part. Assume that there is only one limit point a. Suppose on the contrary that the sequence does not converge to a. We can find some $\varepsilon_0 > 0$ and $n_k \to \infty$ such that $|a_{n_k} - a| \geq \varepsilon_0$. Since $\{a_{n_k}\}$ is bounded, it contains a subsequence $\{a_{n_{k_j}}\}$ which converges to some b satisfying $|b - a| = \lim_{j\to\infty} |a_{n_{k_j}} - a| \geq \varepsilon_0$. Since any subsequence of a subsequence is a subsequence of the original sequence, $\{a_{n_{k_j}}\}$ is again a subsequence of $\{a_n\}$. Hence b is a limit point different from a, contradiction holds.

Let $\{a_n\}$ be a bounded sequence. For each $n \geq 1$, the number

$$\beta_n = \sup_{k\geq n} a_k = \{a_n, a_{n+1}, a_{n+2}, \cdots\} \ ,$$

is a real number. It is clear that $\{\beta_n\}$ is decreasing and bounded from below. By Monotone Convergence Theorem, its limit exists. We call it the **limit superior** of the sequence of $\{a_n\}$.

In notation,

$$\overline{\lim}\ a_n, \text{ or } \limsup\{a_n\} = \lim_{n\to\infty} \beta_n = \inf\{\beta_n\} = \inf_n \sup\{a_k\}_{k\geq n} \ .$$

Similarly, the number

$$\alpha_n = \inf_{k\geq n} a_k = \{a_n, a_{n+1}, a_{n+2}, \cdots\} \ ,$$

is a real number. It is clear that $\{\alpha_n\}$ is increasing and bounded from above. By Monotone Convergence Theorem, its limit exists. We call it the **limit inferior** of the sequence of $\{a_n\}$. In notation,

$$\underline{\lim}\ a_n, \text{ or } \liminf\{a_n\} = \lim_{n\to\infty} \alpha_n = \sup\{\alpha_n\} = \sup_n \inf\{a_k\}_{k\geq n} \ .$$

Theorem 6.2. For a bounded sequence, its supremum is its largest limit point and its infimum the smallest limit point.

The following proof may be skipped in a first reading.

Proof *. Let β be the limit supremum of the bounded sequence $\{a_n\}$ and let $\beta_n = \{a_n, a_{n+1}, a_{n+2}, \cdots, \}$. First we claim that β is greater than or equal to any limit point of $\{a_n\}$. Let $a = \lim_{n_k} a_{n_k}$. For $\varepsilon > 0$, there is some n_{k_0} such that $a - \varepsilon < a_{n_k}$ for all $n_k \geq n_{k_0}$. We have

$$\beta_{n_k} = \sup\{a_{n_k}, a_{n_k+1}, a_{n_k+2}, \cdots\} \geq a_{n_k} \geq a - \varepsilon \ ,$$

for all $n_k \geq n_{k_0}$. As $\beta_n \to \beta$, we can fix an $n_k \geq n_{k_0}$ such that

$$\beta + \varepsilon > \beta_{n_k} \geq a - \varepsilon.$$

We conclude that

$$\beta > a - 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\beta \geq a$.

We claim there is a subsequence converging to β. Since $\beta = \lim_{n\to\infty} \beta_n = \inf_n \beta_n$, for each $N \geq 1$, there is some $n(N)$ such that

$$\beta + \frac{1}{N} > \beta_{n(N)} \geq \beta \ . \tag{1}$$

In other words,

$$\beta + \frac{1}{N} > \sup\{a_{n(N)}, a_{n(N)+1}, a_{n(N)+2}, \cdots\} > \beta - \frac{1}{N} \ .$$

From the definition of the supremum, we can find a_{n_N} from $\{a_{n(N)}, a_{n(N)+1}, a_{n(N)+2}, \cdots\}$ to form a subsequence $\{a_{n_N}\}$ such that

$$\beta_{n(N)} \geq a_{n_N} > \beta_{n(N)} - \frac{1}{N} \ . \tag{2}$$

Combining (1) and (2), we have

$$|a_{n_N} - \beta| < \frac{1}{N} \ .$$

It follows that the subsequence $\{a_{n_N}\}_{N=1}^{\infty}$ converges to β. Similarly, one can treat the case of limit inferior.

Essential 3.5 Cauchy Convergence Criterion

In this section you should know

- Definition of a Cauchy sequence.
- Cauchy Convergence Criterion.
- Some examples (see 3.5.6 in Text, may skip 3.5.7-3.5.11.)

A sequence $\{a_n\}$ is called a **Cauchy sequence** if for each $\varepsilon > 0$, there is some n_ε such that $|a_n - a_m| < \varepsilon, \quad \forall n, m \geq n_\varepsilon$. The main result in this section is

Theorem 6.3 (Cauchy Convergence Criterion). A sequence is convergent if and only if it is a Cauchy sequence.

Proof. $\Rightarrow$. When $\{a_n\}$ converges to a, for $\varepsilon > 0$, there is some n_ε such that $|a_n - a| < \varepsilon/2$ for all $n \geq n_\varepsilon$. By triangle inequality, for $n, m \geq n_\varepsilon$, $|a_n - a_m| \leq |a_n - a| + |a_m - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$, hence $\{a_n\}$ is Cauchy.

$\Leftarrow$. Taking $\varepsilon = 1$, there is some n_1 such that $|a_n - a_m| < 1$ for all $n, m \geq n_1$. It follows that $|a_n| \leq |a_{n_1}| + 1$ which shows that $\{a_n\}$ is a bounded sequence. By Bolzano-Weierstrass Theorem, it has a convergent subsequence $a_{n_k} \to a$. For $\varepsilon > 0$, there is some n_0 such that $|a_{n_k} - a| < \varepsilon/2$ for all $n_k \geq n_0$. On the other hand, as $\{a_n\}$ is Cauchy, for the same ε, there is some n_1 such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq n_1$. Taking $n_2 = \max\{n_0, n_1\}$ and them choosing $m = n_k$, for $n_k \geq n_2$, $|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ for all $n \geq n_2$, hence $\lim_{n\to\infty} a_n = a$ too.

We end our discussion with a remark. The followings are equivalent:

- Order-Completeness Property: Every set bounded from above has a supremum.
- Nested Interval Property: Every nested closed, bounded intervals has non-empty intersection.
- Monotone Convergence Property: Every increasing sequence converges provided it is bounded from above.
- Bolzano-Weierstrass Property: Every bounded sequence has a convergent subsequence.
- Cauchy Completeness Property: Every Cauchy sequence converges.
- Heine-Borel Property: Every covering of $[a, b]$ by open intervals has a finite subcover.

We have learned the first five. A consequence of this equivalence is that one can deduce the other five starting from any one of these properties. I leave the proof to those who are interested. In my school days we used to spend a lot of time in proving all these things, but now it seems to have become out of fashion!